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What is algebra?

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We hear a lot about the importance that all children master algebra before they graduate from high school. But what exactly ***is*** algebra, and is it really as important as everyone claims? And why do so many people find it hard to learn?

Answering these questions turns out to be a lot easier than, well, answering a typical school algebra question, yet surprisingly, few people can give good answers.

First of all, algebra is not "arithmetic with letters." At the most fundamental level, arithmetic and algebra are two different forms of thinking about numerical issues. (I should stress that in this article I'm focusing on school arithmetic and school algebra. Professional mathematicians use both terms to mean something far more general.)

Let's start with arithmetic. This is essentially the use of the four numerical operations addition, subtraction, multiplication, and division to calculate numerical values of various things. It is the oldest part of mathematics, having its origins in Sumeria (primarily today's Iraq) around 10,000 years ago. Sumerian society reached a stage of sophistication that led to the introduction of money as a means to measure an individual's wealth and mediate the exchange of goods and services. The monetary tokens eventually gave way to abstract markings on clay tablets, which we recognize today as the first numerals (symbols for numbers). Over time, those symbols acquired an abstract meaning of their own: *numbers*. In other words, numbers first arose as money, and arithmetic as a means to use money in trade.

It should be noticed that counting predates numbers and arithmetic by many thousands of years. Humans started to count things (most likely family members, animals, seasons, possessions, etc.) at least 35,000 years ago, as evidenced by the discovery of bones with tally marks on them, which anthropologists conclude were notched to provide what we would today call a numerical record. But those early humans did not have numbers, nor is there any evidence of any kind of arithmetic. The tally markers themselves were the record; the marks referred directly to things in the world, not to abstract numbers.

Something else to note is that arithmetic does not have to be done by the manipulation of symbols, the way we are taught today. The modern approach was developed over many centuries, starting in India in the early half of the First Millennium, adopted by the Arabic speaking traders in the second half of the Millennium, and then transported to Europe in the 13th Century. (Hence its present-day name "Hindu-Arabic arithmetic.") Prior to the adoption of symbol-based, Hindu-Arabic arithmetic, traders performed their calculations using a sophisticated system of finger counting or a counting board (a board with lines ruled on it on which small pebbles were moved around). Arithmetic instruction books described how to calculate using words, right up to the 15th Century, when symbol manipulation began to take over.

Many people find arithmetic hard to learn, but most of us succeed, or at least pass the tests, provided we put in enough practice. What makes it possible to learn arithmetic is that the basic building blocks of the subject, numbers, arise naturally in the world around us, when we count things, measure things, buy things, make things, use the telephone, go to the bank, check the baseball scores, etc. Numbers may be abstract — you never saw, felt, heard, or smelled the number 3 — but they are tied closely to all the concrete things in the world we live in.

With ***algebra***, however, you are one more step removed from the everyday world. Those x's and y's that you have to learn to deal with in algebra denote numbers, but usually numbers ***in general***, not particular numbers. And the human brain is not naturally suited to think at that level of abstraction. Doing so requires quite a lot of effort and training.

The important thing to realize is that doing algebra is a **way of thinking** and that it is a way of thinking that is ***different*** from arithmetical thinking. Those formulas and equations, involving all those x's and y's, are merely a way to represent that thinking on paper. They no more are algebra than a page of musical notation is music. It is possible to do algebra without symbols, just as you can play and instrument without being ably to read music. In fact, traders and other people who needed it used algebra for 3,000 years before the symbolic form was introduced in the 16th Century. (That earlier way of doing algebra is nowadays referred to as "rhetorical algebra," to distinguish it from the symbolic approach common today.)

There are several ways to come to an understanding of the difference between arithmetic and (school) algebra.

- First, algebra involves thinking ***logically*** rather than numerically.
- In arithmetic you reason (calculate) ***with*** numbers; in algebra you reason (logically) ***about*** numbers.
- Arithmetic involves ***quantitative*** reasoning with numbers; algebra involves ***qualitative*** reasoning about numbers.
- In arithmetic, you **calculate** a number by working with the numbers you are given; in algebra, you introduce a **term for an unknown number** and reason **logically** to determine its value.

The above distinctions should make it clear that algebra is not doing arithmetic with one or more letters denoting numbers, known or unknown.

For example, putting numerical values for a, b, c in the familiar formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

in order to find the numerical solutions to the quadratic equation

$$ax^2 + bx + c = 0$$

is not algebra, it is arithmetic.

In contrast, deriving that formula in the first place is algebra. So too is solving a quadratic equation not by the formula but by the standard method of "completing the square" and factoring.

When students start to learn algebra, they inevitably try to solve problems by arithmetical thinking. That's a natural thing to do, given all the effort they have put into mastering arithmetic, and at first, when the algebra problems they meet are particularly simple (that's the teacher's classification as "simple"), this approach works.

In fact, the stronger a student is at arithmetic, the further they can progress in algebra using arithmetical thinking. For example, many students can solve the quadratic equation $x^2 = 2x + 15$ using basic arithmetic, using no algebra at all.

Paradoxically, or so it may seem, however, those better students may find it harder to learn algebra. Because to do algebra, for all but the most basic examples, you have to ***stop*** thinking arithmetically and learn to think algebraically.

Is mastery of algebra (i.e., algebraic thinking) worth the effort? You bet — though you'd be hard pressed to reach that conclusion based on what you will find in most school algebra textbooks. In today's world, most of us really do need to master algebraic thinking. In particular, you need to use algebraic thinking if you want to write a macro to calculate the cells in a spreadsheet like Microsoft *Excel*. This one example alone makes it clear why algebra, ***and not arithmetic***, should now be the main goal of school mathematics instruction. With a spreadsheet, you don't need to do the arithmetic; the computer does it, generally much faster and with greater accuracy than any human can. What you, the person, have to do is create that spreadsheet in the first place. The computer can't do that for you.

It doesn't matter whether the spreadsheet is for calculating scores in a sporting competition, keeping track of your finances, running a business or a club, or figuring out the best way to equip your character in *World of Warcraft*, you need to think ***algebraically*** to set it up to do what you want. That means thinking about or across numbers ***in general***, rather than in terms of (specific) numbers.

Of course, the need for algebra does not make it any easier to learn — though I think that spreadsheets can provide today's students with more meaningful and fulfilling applications than problems about trains leaving stations or garden hoses filling swimming pools, that my generation had to endure. But in a world where our very national livelihood depends on staying ahead of the technology curve, it is crucial that we equip our students with the kind of thinking skills today's world requires. Being able to use computers is one of those skills. And being able to use a computer to do arithmetic requires ***algebraic*** thinking.

In future [illegible]'ll describe the growth of algebra through the ages.